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Time-Frequency and Time-Scale Vector Fields for Deforming Time-Frequency and Time-Scale Representations

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ABSTRACT

We study local deformations of time-frequency and time-scale representations, in the framework of the so-called reassignment methods, which aim at "deblurring" time-frequency representations. We focus on deformations generated by appropriate vector fields defined on time-frequency or time scale plane, and constructed on the basis of geometric and group-theoretical arguments. Such vector fields may be used as such for signal analysis (as quantities generalizing instantaneous frequency or group delay) in the framework of reassignment algorithms.

Keywords: Time-Frequency Analysis, Gabor transform, Wavelets, instantaneous frequency, group delay, reassignment, group representation, Bargmann space

1. INTRODUCTION

Linear time-frequency representations such as the continuous wavelet or Gabor transforms are sometimes considered as time-frequency representations with poor resolution. This comes from the fact that their square modulus may be seen as particular smoothings of Wigner functions, which are known to have sharp localization properties (at least for some particular classes of signals). The problem of "de-blurring" such linear representations has received some attention since the early attempt by Kodera, Gendrin and DeVilledary,[1] revisited more recently by Auger and Flandrin.[2] More recent contributions, due to Chassande-Mottin et al,[3,4] and Daubechies and Maes[5] should also be quoted.

Reassignment methods may be seen as an attempt to restore the resolution (in time-frequency or time-scale space) which has been lost when using wavelet or Gabor transforms rather than supposedly "optimal" bilinear transforms. They aim at "sharpening" a given time-frequency representation by "deforming" it, i.e. by shifting coefficients in the time-frequency plane using an appropriate prescription for the shifts. The main idea is the following. Let $\mathcal{R} : \mathbb{R}^2 \to \mathbb{R}^2$ be a map on the time-frequency plane, called the *reassignment map*. To any time-frequency representation $\rho \in L^1(\mathbb{R}^2)$, associate the measure on $\mathbb{R}^2$ defined as follows: for any $\Omega \subset \mathbb{R}^2$, denote by $\mathcal{R}^{-1}\Omega$ the inverse image of Ω by $\mathcal{R}$, i.e the set $\mathcal{R}^{-1}\Omega = \left\{(b,\nu) \in \mathbb{R}^2, \mathcal{R}(b,\nu) \in \Omega\right\}$. Then set

$$M(\Omega) = \int_{\mathcal{R}^{-1}\Omega} \rho(b,\nu) db d\nu \ . \tag{1}$$

By construction, the total mass of ρ is preserved, which is an important property.

The main issue is the choice of the reassignment map. When ρ is the square modulus of a continuous Gabor transform (see below), a prescription has been given by Kodera, Gendrin and de Villedary,[1] which was reconsidered and studied later on by Auger and Flandrin.[2] In such a situation, the time-frequency representation under consideration is such that its total mass equals (up to a constant) the energy of the signal, and also equals the total mass of the reassigned representation. However, a continuous Gabor transform is a complex valued function, and the reassignment does not take at all into account its argument (even though the argument is used explicitely for computing the reassignment map). Also, since only the spectrogram is reassigned (and not the CGT), the method does not provide any way of reconstructing the original signal from its reassigned time-frequency representation.

An alternative has been proposed by Daubechies and Maes,[5] based on the continuous wavelet transform. In that approach, the reassignment is performed on the (generally complex valued) transform itself, rather than on its square modulus. This yields the so-called *synchrosqueezed wavelet transform*. A main feature of that transform is that, in contrast with the other reassigned transforms, it is invertible: the signal may be reconstructed from its synchrosqueezed wavelet transform. However, let us mention that the corresponding reassignment map does not handle the argument of the wavelet transform in a natural way, in the sense that the sum in Eq. (1) involves complex numbers with different arguments. We shall come back to that point later on.

More recently, E. Chassande-Mottin and his collaborators[3, 4] suggested a different interpretation of the reassignment maps, in terms of infinitesimal displacements of coefficients of the considered time-frequency representation. They proposed to use reassignment maps for building a vector field on the time-frequency plane, and use such a vector field as a starting point for further signal processing tasks. Such an approach has found several interesting applications, for example for time-frequency partition of signals. As we shall see, such an approach is also quite natural when more theoretical and abstract arguments are taken into account.

We propose in this paper a different interpretation of reassignment and reassignment maps, very much in the spirit of the differential reassignment of Refs.[3] and[4]. Our first remark is the fact that time-frequency representations are generally functions defined on a two-dimensional space, called the *phase space*, or the *time-frequency space*, which turns out to be an homogeneous space for some motion group: the Weyl-Heisenberg group in the Gabor case, and the affine group in the wavelet case. In both cases, the group action plays the role of translations on the time-frequency space. Therefore, it makes sense to use that particular group action for "displacing" coefficients on the phase space and for designing reassignment algorithms.

More precisely, we consider displacements on the phase space generated by vector fields, satisfying two constraints. The first constraint is a covariance constraint: covariance with respect to the underlying group action. This is an important constraint, for it ensures that the reassigned time-frequency transform will share the covariance properties of the initial transform. The second constraint is phase invariance: we impose that the argument of the time-frequency transform be unchanged by the deformation. Such a constraint originates from more "practical considerations": it has been recognized for a long time that the phase of continuous time-frequency transform may be of great importance for many tasks of signal analysis and processing.[6–8] We shall prove that such requirements are sufficient to yield the classical prescription[1] for Gabor reassignment, and a new one for the wavelet reassignment.

The paper is organized as follows. We set up our notations and describe general results in Section 2. Then we describe the case of Gabor transform and the Weyl-Heisenberg group in Section 3, and the wavelet case in Section 4. Section 5 is devoted to conclusions.

2. PRELIMINARIES

Throughout this paper, we shall mainly be concerned with the space of finite energy signals $L^2(\mathbb{R})$, equipped with the scalar product

$$\langle f, g\rangle = \int f(t)\overline{g}(t)\, dt\ .$$

Among the time-frequency representations, the so-called Wigner (or Wigner-Ville) distribution (the WVD) plays a prominent role. The WVD is defined as follows. For $f \in L^2(\mathbb{R})$, one sets

$$\mathcal{W}_f(b,\nu) = \int f\left(b+\frac{t}{2}\right)\overline{f}\left(b-\frac{t}{2}\right)e^{-2i\pi\nu t}\, dt\ . \tag{2}$$

This defines a bounded continuous function of b and ν, which in addition belongs to all the spaces $L^p(\mathbb{R}^2)$, $1 \le p \le \infty$ (see Ref.[9] for a proof). In particular, it is well known that

$$\int_{\mathbb{R}^2} \mathcal{W}_f(b,\nu)\ db\, d\nu = ||f||^2\ , \tag{3}$$

which suggests to interpret $\mathcal{W}_f(b,\nu)/||f||^2$ as a probability density in the two-dimensional plane. The WVD also satisfies the following marginal properties:

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{W}_f(b,\nu)\ db &= |\hat{f}(\nu)|^2\ , && (4)\\ \int_{\mathbb{R}} \mathcal{W}_f(b,\nu)\ d\nu &= |f(b)|^2\ . && (5) \end{aligned}$$

This in turn also suggests to introduce average quantities: namely, the instantaneous frequency

$$\tilde{\nu}(b) = \frac{1}{|Z_f(b)|^2} \int \nu \mathcal{W}_{Z_f}(b,\nu)\, d\nu \ , \tag{6}$$

where Z_f is the analytic signal* of f. The instantaneous frequency provides an estimate for a "local frequency" of the signal†. Similarly, the group delay may be defined as another appropriate moment of the Wigner-Ville distribution:

$$\tilde{\tau}(\nu) = \frac{1}{|\widehat{Z_f}(\nu)|^2} \int b \mathcal{W}_{Z_f}(b,\nu)\, db \ . \tag{7}$$

The notion of instantaneous frequency has been criticized on the basis of physical arguments: its physical meaning can be rather questionable for complex signals. Indeed, suppose that a fonction f is the sum of two sine waves. Then its instantaneous frequency is an oscillatory function, which does carry information about the signal, but does not reflect its very nature.

In such situations one would need to introduce several such instantaneous frequencies, which would correspond to different frequency ranges. Such a program may be realized in several ways, but is most often based on other time-frequency representations. For example, given a window $g \in L^2(\mathbb{R})$, and letting $G_f(b,\nu)$ denote the continuous Gabor transform of $f \in L^2(\mathbb{R})$ (details will be given in Section 3 below), one may introduce new quantities:

$$\begin{aligned} \tilde{b}(b,\nu) &= \frac{1}{|G_f(b,\nu)|^2} \int \tau \mathcal{W}_f(\tau,\xi)\, \mathcal{W}_g(\tau-b,\xi-\nu) d\tau d\xi \ , & (8) \\ \tilde{\nu}(b,\nu) &= \frac{1}{|G_f(b,\nu)|^2} \int \xi \mathcal{W}_f(\tau,\xi)\, \mathcal{W}_g(\tau-b,\xi-\nu) d\tau d\xi \ , & (9) \end{aligned}$$

which indeed provide estimates for time-frequency dependent notions of time and frequency. It is interesting to notice the similarity of the expressions defining $\tilde{\nu}(b)$ and $\tilde{\tau}(\nu)$ on one hand, and $\tilde{b}(b,\nu)$ and $\tilde{\nu}(b,\nu)$ on the other hand. In both case, these quantities arise as mean values of time and frequency variables, with respect to an appropriate "density", thus defined by an "integral" formula. However, the instantaneous frequency and the group delay also arise from a "differential formula", expressed as a logarithmic derivative of a Gabor transform. We shall show below that $\tilde{b}(b,\nu)$ and $\tilde{\nu}(b,\nu)$ may also be defined via a logarithmic derivative, for an appropriately chosen derivative.

3. DEFORMATION OF CONTINUOUS GABOR TRANSFORM

3.1. Generalities

To fix the notations, let us start by describing some of the main features of the continuous Gabor transform (*CGT* for short). More details may be found in many textbooks, for example Ref.[10] for group theoretical aspects, or Ref.[11] for signal processing applications.

Definition 1. Let $g \in L^2(\mathbb{R})$, $g \neq 0$. The continuous Gabor transform of any finite energy signal $f \in L^2(\mathbb{R})$ is the function of two variables

$$G_f(b,\nu) = \langle f, g_{(b,\nu)} \rangle \ , \tag{10}$$

where $b,\nu \in \mathbb{R}$, and the Gaborlets $g_{(b,\nu)}$ are shifted and modulated copies of the reference window g, defined by

$$g_{(b,\nu)}(t) = e^{2in\pi\nu(t-b)} g(t-b) \tag{11}$$

*Given $f \in L^2(\mathbb{R})$, its analytic signal is the function $Z_f \in L^2(\mathbb{R})$ such that $\widehat{Z_f}(\nu) = 2\hat{f}(\nu)\,\Theta(\nu)$, Θ being Heaviside's step function.

†Because of Heisenberg's inequality, it is extremely difficult to make sense to such a notion. However, from the intuitive point of view, the local frequency is expected to make sense for signals to which local amplitude and phase may be associated, in such a way that the amplitude varies on scales much larger than the oscillations coming from the phase.

Here, n is an arbitrary positive integer.

In general, one chooses $n = 1$, but we keep it arbitrary for the time being. Notice that our convention differs from that used in Ref.[3], where the symmetrized version of the CGT[‡] is used. Both formulations may be obtained from each other modulo minor modifications. It is well known that after a suitable normalization, the Gabor transform is an isometry of $L^2(\mathbb{R})$ to the space $L^2(\mathbb{R}^2)$ (unless otherwise stated, the measure is the Lebesgue measure): for all $f \in L^2(\mathbb{R})$, $G_f \in L^2(\mathbb{R}^2)$, and one has the norm equivalence (provided $||g|| \neq 0$!)

$$\int |G_f(b,\nu)|^2 \, dbd\nu = ||g||^2 \int |f(t)|^2 \, dt \ . \tag{12}$$

Therefore, it may be inverted by its adjoint: for all $f \in L^2(\mathbb{R})$, we may write

$$f = \frac{1}{||g||^2} \int G_f(b,\nu) \; g_{(b,\nu)} \, dbd\nu \ , \tag{13}$$

the equality being understood in the weak L^2-sense, i.e. in the sense of scalar products with $L^2(\mathbb{R})$ functions.

The square modulus of the CGT (called the *spectrogram*) enjoys interesting properties. We emphasize two of them below:

1. For $f \in L^2(\mathbb{R})$, $|G_f|^2$ is a smoothed version of the Wigner transform of f:

$$|G_f(b,\nu)|^2 = \int \mathcal{W}_f(\tau,\xi) \, \mathcal{W}_g(\tau - b, \xi - \nu) \, d\tau d\xi \ . \tag{14}$$

2. The spectrogram is invariant with respect to time and frequency translations: given $f \in L^2(\mathbb{R})$, let $f' \in L^2(\mathbb{R})$ be defined by $f'(t) = e^{2in\pi\lambda t} f(t-\tau)$, where λ, τ are fixed real numbers. Then it is easily verified that $|G_{f'}(b,\nu)|^2 = |G_f(b-\tau, \nu-\lambda)|^2$.

However, the CGT itself is *not* fully covariant with respect to shifts and modulations. Indeed, if again $f'(t) = e^{2in\pi\lambda t} f(t-\tau)$, we have

$$G_{f'}(b,\nu) = e^{2in\pi\lambda(b-\tau)} G_f(b-\tau, \nu-\lambda) \ . \tag{15}$$

This is intimately related to the fact that time and frequency translations do not form a group. They in fact generate a three-dimensional group, called the Weyl-Heisenberg group.

3.2. The Weyl-Heisenberg group, and its regular representations

We first give a brief account of the harmonic analysis on the Weyl-Heisenberg group. The interested reader is invited to consult Refs.[10] and[12] for more details.

The Weyl-Heisenberg group is of the form $G_{WH} = \mathbb{R}^2 \times [0,1[$, with group law

$$(b,\nu,\varphi) \cdot (b',\nu',\varphi') = (b+b', \nu+\nu', \varphi+\varphi'+\nu b') \ , \tag{16}$$

and may be identified with a group of matrices

$$(b,\nu,\varphi) \approx \begin{pmatrix} 1 & \nu & \varphi \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \ .$$

The unit element is $(0,0,0)$, and the inverse of any $(b,\nu,\varphi) \in G_{WH}$ is given by

$$(b,\nu,\varphi)^{-1} = (-b, -\nu, -\varphi + b\nu) \ . \tag{17}$$

[‡]defined by $G_f(b,\nu) = \int f(t) e^{-2i\pi\nu(t-b/2)} \overline{g}(t-b) \, dt$

The Weyl-Heisenberg group is unimodular: the Haar measure μ, defined by

$$d\mu(b,\nu,\varphi) = db\, d\nu\, d\varphi \tag{18}$$

is both left and right invariant: for any $(b_0, \nu_0, \varphi_0) \in G_{WH}$,

$$d\mu((b_0,\nu_0,\varphi_0)(b,\nu,\varphi)) = d\mu((b,\nu,\varphi)(b_0,\nu_0,\varphi_0)) = d\mu(b,\nu,\varphi) \ .$$

The complete classification of irreducible unitary representations of the Weyl-Heisenberg group is a classical result of harmonic analysis on Lie groups.[10] Such representations are essentially characterized by an integral number $n \in \mathbb{Z}$. More precisely, except for the characters, all of them act on $L^2(\mathbb{R})$, as follows: if $f \in L^2(\mathbb{R})$,

$$[\pi_n(b,\nu,\varphi)f](t) = e^{2i\pi n(\varphi+\nu(t-b))} f(t-b) \ . \tag{19}$$

Therefore, we have

$$g_{(b,\nu)}(t) = \pi_n(b,\nu,0)g(t) \ . \tag{20}$$

Let $L^2(G_{WH})$ denote the space of functions on G_{WH} which are square-integrable with respect to the invariant[§] measure $d\mu(b,\nu,\varphi) = db\, d\nu\, d\varphi$. A main aspect of harmonic analysis on the Weyl-Heisenberg group is the study of the so-called *regular representations* of G_{WH} on $L^2(G_{WH})$

The left regular representation The left-regular representation λ of G_{WH} on $L^2(G_{WH})$ is defined as follows: if $F \in L^2(G_{WH})$,

$$\lambda(g)F(h) = F(g^{-1}h) \ . \tag{21}$$

In explicit form, we then have

$$\lambda(b',\nu'\varphi')F(b,\nu,\varphi) = F(b-b',\nu-\nu',\varphi-\varphi'+\nu'(b'-b)) \ . \tag{22}$$

λ is a unitary representation, and is reducible. Its reduction goes as follows. Introduce the family of spaces

$$\begin{aligned}\mathcal{H}_n \quad &= \Big\{ F \in L^2(G_{WH}), F((b,\nu,\varphi)(0,0,\varphi')) = F(b,\nu,\varphi)e^{-2i\pi n\varphi'} \\ &\qquad\qquad \text{for all } (b,\nu,\varphi)(0,0,\varphi') \in G_{WH} \Big\} \ .\end{aligned} \tag{23}$$

Note that if $F \in \mathcal{H}_n$, then one has $F(b,\nu,\varphi) = F(b,\nu,0)e^{-2i\pi n\varphi} := F_n(b,\nu)e^{-2i\pi n\varphi}$. Therefore, $\mathcal{H}_n \simeq L^2(\mathbb{P}) \simeq L^2(\mathbb{R}^2)$, where $\mathbb{P} = G_{WH}/[0,1[\simeq \mathbb{R}^2$.

The connection between the continuous Gabor transform (as well as the continuous wavelet transform) and the theory of square-integrable group representation has been observed and studied in details in Refs.[13] and[14].

Let now $F \in L^2(G_{WH})$. Using a Fourier series decomposition with respect to the variable φ, we write

$$F(b,\nu,\varphi) = \sum_n F_n(b,\nu)e^{-2in\pi\varphi} \ . \tag{24}$$

Therefore, $L^2(G_{WH})$ splits into a direct sum

$$L^2(G_{WH}) = \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_n \ , \tag{25}$$

The group acts onto $\mathcal{H}_n$ as follows: if $F_n \in \mathcal{H}_n$ and $(b',\nu',\varphi') \in G_{WH}$,

$$\lambda(b',\nu',\varphi')F_n(b,\nu) = e^{2in\pi\nu'(b-b')}e^{2in\pi\varphi'}F_n(b-b',\nu-\nu') \ . \tag{26}$$

[§]in the sense $d\mu\left((b',\nu',\varphi')(b,\nu,\varphi)\right) = d\mu(b,\nu,\varphi)$ for all $(b',\nu',\varphi') \in G_{WH}$.

From now on, we keep the index n fixed, and we denote by λ_n the map $\mathbb{R}^2 \to \mathcal{U}(L^2(\mathbb{R}^2))$ defined by:

$$[\lambda_n(b',\nu')F](b,\nu) = e^{2in\pi\nu'(b-b')} F(b-b',\nu-\nu') \ . \tag{27}$$

Remark: The representation λ_n realizes the translations on $\mathbb{P} \simeq \mathbb{R}^2$, viewed as an homogeneous space for the action of the Weyl-Heisenberg group. The space $\mathbb{P}$, endowed with the action of G_{WH} above, is called *phase space*, or *Time-Frequency plane*[¶]. Hence, translations of functions on the phase space differ from the usual translations, in the sense that the shift is accompanied by a phase correction.

The following property is a direct consequences of the above definitions.

Proposition 1. Let $g \in L^2(\mathbb{R})$, $g \neq 0$. Let $f \in L^2(\mathbb{R})$, and let G_f denote its CGT, as defined in Definition 1. Then $G_f \in \mathcal{H}_n$. In addition, we have, for all $h = (b',\nu',0) \in G_{WH}$

$$G_{\pi_n(h)f} = \lambda_n(b',\nu')G_f \ . \tag{28}$$

The infinitesimal generators of λ_n are also of interest: introduce the differential operator $\check{\nabla} = (\check{\partial}_b, \check{\partial}_\nu)^t$, where for any smooth enough function F, we set

$$\begin{aligned} \check{\partial}_b F(b,\nu) &= \partial_b F(b,\nu) && (29)\\ \check{\partial}_\nu F(b,\nu) &= \partial_\nu F(b,\nu) - 2in\pi b F(b,\nu) \ . && (30) \end{aligned}$$

Then we have that for all $F \in L^2(\mathbb{R}^2)$,

$$[\lambda_n(v_b,v_\nu)F](b,\nu) = e^{-v_b\check{\partial}_b} e^{-v_\nu\check{\partial}_\nu} F(b,\nu) \ . \tag{31}$$

When F is an analytic function, the latter equality is proved using infinite Taylor series expansion for F and the exponentials. The extension to $L^2(\mathbb{P})$ uses standard density arguments.

The right regular representation Similarly, one introduces the right regular representation ρ, defined by

$$\rho(g)F(h) = F(hg) \ . \tag{32}$$

The right regular representation leave the $\mathcal{H}_n$ spaces invariant, and one may define

$$\rho_n(b',\nu')F_n(b,\nu) = e^{-2in\pi\nu b'} F(b+b',\nu+\nu') \tag{33}$$

Again, the infinitesimal generators of the right regular representation are of interest. As before, we introduce the modified derivative $\hat{\nabla} = (\hat{\partial}_b, \hat{\partial}_\nu)^t$, where for any smooth enough function F, we set

$$\begin{aligned} \hat{\partial}_b F(b,\nu) &= \partial_b F(b,\nu) - 2in\pi\nu F(b,\nu) && (34)\\ \hat{\partial}_\nu F(b,\nu) &= \partial_\nu F(b,\nu) \ . && (35) \end{aligned}$$

Then we have that for all $F \in L^2(\mathbb{R}^2)$,

$$[\rho_n(v_b,v_\nu)F](b,\nu) = e^{v_b\hat{\partial}_b} e^{v_\nu\hat{\partial}_\nu} F(b,\nu) \ . \tag{36}$$

[¶]In technical terms, $\mathbb{P}$ is a symplectic manifold (with symplectic form $[(b,\nu),(b',\nu')] = b\nu' - \nu b'$).

Properties The above derivatives enjoy interesting properties (in particular some covariance properties), which we summarize in the following proposition:

Proposition 2.

1. The infinitesimal generators satisfy the following commutation relations:
$$[\check{\partial}_b, \check{\partial}_\nu] = -2in\pi \ , \quad [\hat{\partial}_b, \hat{\partial}_\nu] = 2in\pi \ ,$$
2. One has the following covariance properties: for all F, and for all $(b,\nu) \in \mathbb{R}^2$,
$$\widehat{\nabla}(\lambda_n(b,\nu)F) = \lambda_n(b,\nu)\widehat{\nabla}F \ , \quad \check{\nabla}(\rho_n(b,\nu)F) = \rho_n(b,\nu)\check{\nabla}F \ . \tag{37}$$

Sketch of the proof: The first assertion follows from a direct calculation. For the second one, we first remark that, as a consequence of the associativity of the group law, the left and right regular representations λ and ρ commute with each other. Therefore, they also commute with the infinitesimal generator of one another, which proves the second assertion of the proposition.

3.3. Constant Phase Deformations

Let us now return to the problem of deforming a CGT. The goal of this section is to provide a geometrical interpretation of the reassignment maps proposed and studied in Refs.[1] and[2] for the CGT. Let us first set the problem. We are interested in finding a vector field

$$\underline{v} : (b,\nu) \to \underline{v}(b,\nu) = (v_b(b,\nu), v_\nu(b,\nu))^t \in \mathbb{R}^2$$

defined on the phase space, such that the argument of the CGT remains unchanged under the infinitesimal action of the Weyl-Heisenberg group. More precisely, we ask for invariance with respect to the right regular action $\rho_n(\epsilon\underline{v})$, at first order in $\epsilon > 0$. The reasons of the choice of the right-regular representation ρ rather than λ will be given in subsection 3.5 below.

Let F be a smooth function defined on the plane. For $\epsilon > 0$ and $\underline{v} \in \mathbb{R}^2$, let $\tau_{\underline{v}}(\epsilon) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ denote the operator defined by

$$\tau_{\underline{v}}(\epsilon)F(b,\nu) = \rho_n(-\epsilon\underline{v}(b,\nu))\, F(b,\nu) \ . \tag{38}$$

At first order in ϵ, we have

$$\tau_{\underline{v}}(\epsilon)F(b,\nu) = F(b,\nu) - \epsilon\underline{v}(b,\nu) \cdot \widehat{\nabla}F(b,\nu) + O(\epsilon^2) \ , \tag{39}$$

where the symbol "$\cdot$" stands for the scalar product in $\mathbb{C}^2$.

Let now $f \in L^2(\mathbb{R})$, and let G_f denote its CGT. Let us write

$$G_f(b,\nu) = |G_f(b,\nu)|\, e^{i\Omega(b,\nu)} \ , \tag{40}$$

where $\Omega(b,\nu) = \arg(G_f(b,\nu))$. One immediately sees that, again at first order in ϵ,

$$\tau_{\underline{v}}(\epsilon)G_f(b,\nu) = G_f(b,\nu) - \epsilon G_f(b,\nu)\left(\underline{v} \cdot \frac{|\nabla G_f(b,\nu)|}{|G_f(b,\nu)|} + i\underline{v} \cdot \nabla^m\Omega(b,\nu)\right) + O(\epsilon^2) \ , \tag{41}$$

where we have introduced the *modified gradient* ∇^m, defined by

$$\nabla^m\Omega(b,\nu) = \begin{pmatrix} \partial_b^m\Omega(b,\nu) \\ \partial_\nu^m\Omega(b,\nu) \end{pmatrix} = \begin{pmatrix} \partial_b\Omega(b,\nu) - 2n\pi\nu \\ \partial_\nu\Omega(b,\nu) \end{pmatrix} \tag{42}$$

Therefore, imposing that the argument of $G_f(b,\nu)$ be unchanged imposes that the vector field $\underline{v}$ be of the form

$$\underline{v}(b,\nu) = \frac{\sigma}{2\pi}\begin{pmatrix} \partial_\nu^m\Omega(b,\nu) \\ -\partial_b^m\Omega(b,\nu) \end{pmatrix} = \frac{\sigma}{2\pi}\begin{pmatrix} \partial_\nu\Omega(b,\nu) \\ -\partial_b\Omega(b,\nu) + 2n\pi\nu \end{pmatrix} \ , \tag{43}$$

where $\sigma = \sigma(b,\nu) \in \mathbb{R}$ is a real number. For the sake of simplicity, we shall assume that the sign of σ is constant.

3.4. Gaussian windows and the Bargmann space

Up to now, we have essentially focused on the study of the argument of the transform, and imposed a "constant phase" constraint. We now focus on the study of the modulus. Let $(b,\nu) \to \underline{v}(b,\nu)$ be a time-frequency vector field as in (43). Then we have, at first order in ϵ,

$$\tau_{\underline{v}}(\epsilon)G_f(b,\nu) = G_f(b,\nu) - \epsilon G_f(b,\nu)p(b,\nu) + O(\epsilon^2) \ , \tag{44}$$

where we have introduced the quantity

$$p(b,\nu) = \underline{v} \cdot \frac{|\nabla G_f(b,\nu)|}{|G_f(b,\nu)|} = \frac{\sigma}{2\pi} \begin{pmatrix} \partial_\nu \Omega(b,\nu) \\ -\partial_b \Omega(b,\nu) + 2n\pi\nu \end{pmatrix} \cdot \frac{|\nabla G_f(b,\nu)|}{|G_f(b,\nu)|} \ , \tag{45}$$

We start with the following observation. Let us introduce the complex variable $z = \nu + ib$, and set

$$\hat{\partial}_{\overline{z}} = \hat{\partial}_\nu + i\hat{\partial}_b \tag{46}$$

Then we have

> **Proposition 3.** Assume that the sign of $\sigma(b,\nu)$ is independent of b and ν. A sufficient condition for the sign of $p(b,\nu)$ in (45) to be constant is that the CGT G_f satisfy the Cauchy-type equation
>
> $$\hat{\partial}_{\overline{z}} G_f(b,\nu) = 0 \tag{47}$$
>
> Such a condition is automatically satisfied for Gaussian windows:
>
> $$g(t) = Ce^{-\pi n t^2} \ , \tag{48}$$
>
> where $C \in \mathbb{C}$ is a constant.

The meaning of the proposition is the following: if the window g satisfies condition (47), imposing a condition of constant phase of the "displaced" transform ensures that the displacement always goes globally in the direction of increasing or decrasing values of $|G_f|$.

Proof of the proposition: Let us assume that (47) holds, and observe that

$$\Re\left(\frac{\hat{\partial}_b G_f}{G_f}\right) = \frac{\partial_b |G_f|}{|G_f|} \ , \quad \text{and} \quad \Re\left(\frac{\hat{\partial}_\nu G_f}{G_f}\right) = \frac{\partial_\nu |G_f|}{|G_f|} \ .$$

$$\Im\left(\frac{\hat{\partial}_b G_f}{G_f}\right) = \partial_b^m \Omega \ , \quad \text{and} \quad \Im\left(\frac{\hat{\partial}_\nu G_f}{G_f}\right) = \partial_\nu^m \Omega \ .$$

Then,

$$p(b,\nu) = \sigma\left(\partial_\nu^m \Omega \frac{\partial_b |G_f|}{|G_f|} - \partial_b^m \Omega \frac{\partial_\nu |G_f|}{|G_f|}\right) = -\sigma \frac{|\hat{\partial}_b G_f|^2}{|G_f|^2} \ .$$

Therefore, a positive value of σ ensures that the action of $\tau_{\underline{v}}$ increases the modulus of G_f.

Let us now study in more details the solutions of the Cauchy-type equation (47). Let us express $G_f(b,\nu)$ in the form

$$G_f(b,\nu) = F(z,\overline{z}) \ . \tag{49}$$

Then, equation (47) is equivalent to

$$\partial_{\overline{z}} F(z,\overline{z}) = \frac{\pi}{2} n(z+\overline{z}) F(z,\overline{z}) \ ,$$

whose solutions are of the form

$$F(z,\overline{z}) = \mathcal{F}(z) e^{-\frac{\pi}{4} n(z+\overline{z})^2} \ ,$$

where $\mathcal{F}(z)$ is a holomorphic function. Simple manipulations finally yield the following expression, reminiscent of the *Bargmann factorization*[15]

$$G_f(b,\nu) = \mathcal{G}_f(z)\, e^{-\frac{\pi}{2}n(|z|^2+\frac{\bar{z}^2}{2})} \ , \tag{50}$$

where $\mathcal{G}_f(z)$ is a holomorphic function of $z = \nu + ib$.

Let us now consider a window g of the form (48). Then, an explicit computation yields

$$G_f(b,\nu) = \left(e^{\frac{3\pi}{4}nz} \int f(t) e^{-2in\pi tz} e^{-\pi nt^2}\, dt \right)\ e^{-\frac{\pi}{2}n(|z|^2+\frac{\bar{z}^2}{2})} \ , \tag{51}$$

i.e. precisely of the form (50). This completes the proof of the proposition.

For completeness, let us notice that the Bargmann factorization obtained here assumes a form slightly different from the form obtained when using the symmetrized version of the CGT. We refer to Ref.[4] for a detailed presentation of the symmetrized case.

3.5. Application to reassignment

Let us now come back to the reassignment problem. Our goal here is to set up a rationale for obtaining a prescription for reassignment maps, from first principles. For the sake of simplicity, we assume that σ is a (positive) constant, and we set it to 1 (without loss of generality, as σ can be absorbed in the constant ϵ). Our approach emphasizes two constraints:

1. *Covariance:* Let $f \in L^2(\mathbb{R})$ and $x \in G_{WH}$; let $f' = \pi_n(x)f$. With the same notations as before, we know from Proposition 1 that $G_{f'} = \lambda_n(x)G_f$. Therefore, imposing covariance of the reassignment with respect to the action of G_{WH} on $L^2(\mathbb{R})$ implies that one must use translations on $\mathbb{P}$ which commute with λ_n, i.e. the representation ρ_n.

2. *Constant phase:* this constraint ensures that at first order, the reassignment preserves phase coherence.

These two constraints lead us to the following prescription for (generalized) infinitesimal reassignment:

$$G_f(b,\nu) \longmapsto \rho_n(-\epsilon \underline{v}(b,\nu)) G_f(b,\nu) = e^{2in\pi\epsilon\nu v_b(b,\nu)} G_f(b - \epsilon v_b(b,\nu), \nu - \epsilon v_\nu(b,\nu)) \ , \tag{52}$$

with $\underline{v} = \underline{v}(b,\nu)$ as in Equation (43) and $\sigma = 1$. We notice in particular that if we do not take the arguments into account, or equivalently, if we limit ourselves to the reassignment of the spectrogram, the prescription

$$\begin{cases} b & \to b - \epsilon v_b(b,\nu) \quad = b - \frac{\epsilon}{2\pi}\partial_\nu \Omega(b,\nu) \\ \nu & \to \nu - \epsilon v_\nu(b,\nu) \quad = \nu(1-n\epsilon) + \frac{\epsilon}{2\pi}\partial_b \Omega(b,\nu) \end{cases} \tag{53}$$

coincides[‖] with the prescription given by E. Chassande-Mottin and his collaborators,[3,4] in the framework of the *differential reassignment.*

Remark: We notice that the choice $\epsilon = 1/n$ yields the reassignment maps originally proposed in Ref.[1] (where only the case $n = 1$ is considered):

$$\begin{cases} b & \to \tilde{b} \quad = b - \dfrac{1}{2n\pi}\, \partial_\nu \Omega(b,\nu) \\ \nu & \to \tilde{\nu} \quad = \dfrac{1}{2n\pi}\, \partial_b \Omega(b,\nu) \end{cases} \tag{54}$$

and also obtained in terms of the Wigner function. At the present stage of our work, we have no explanation for this coincidence, since our point of view is that of infinitesimal reassignment.

[‖] up to the modifications needed for replacing the CGT used here with the symmetrical one used in Ref.[4].

Remark: We also notice that for particular choice $\epsilon = 1/n$, we obtain directly the new locations $\tilde{b}$ and $\tilde{\nu}$ in terms of the infinitesimal generator of the left representation λ:

$$\left\{ \begin{array}{lcl} \tilde{b}(b,\nu) & = & -\dfrac{1}{2n\pi}\,\Im\dfrac{\check{\partial}_\nu G_f(b,\nu)}{G_f(b,\nu)} \\ \tilde{\nu}(b,\nu) & = & \dfrac{1}{2n\pi}\,\Im\dfrac{\check{\partial}_b G_f(b,\nu)}{G_f(b,\nu)} \end{array} \right. \tag{55}$$

This fact is interesting by itself, as it shows that the prescription $(b,\nu) \to (\tilde{b},\tilde{\nu})$ is covariant with respect to the left translations λ on the time-frequency space, therefore with respect to the action of G_{WH} on $L^2(\mathbb{R})$.

Such formulas are reminiscent of the formulas which define the instantaneous frequency and the group delay in terms of logarithmic derivatives of the analytic signal: for example

$$\tilde{\nu}(t) = \frac{1}{2\pi}\,\frac{\partial_t Z_f(t)}{Z_f(t)}\ ,$$

and

$$\tilde{\tau}(\nu) = -\frac{1}{2\pi}\,\frac{\partial_\nu \hat{Z}_f(\nu)}{\hat{Z}_f(\nu)}\ ,$$

(we recall that Z_f is the analytic signal of f). Again, the expressions obtained here are similar, and in both cases the derivative used is covariant with respect to the underlying group action (usual translations on the real line, and left-translations on the time-frequency space).

4. THE CASE OF WAVELET TRANSFORM

The case of the continuous wavelet transform (CWT) is slightly different from that of the CGT, because it is based upon a group with a different structure. However, the main features of the analysis are similar. In particular, one still has a well-defined notion of phase space $\mathbb{P}$, and of left and right regular representations of the underlying group G_{aff} on $L^2(\mathbb{P})$.

4.1. The continuous wavelet transform

Let us start by recalling the basic aspects of continuous wavelet analysis. We shall mainly limit ourselves to the space of analytic signals, i.e. the Hardy space

$$H^2_+(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}),\ \hat{f}(\omega) = 0\,\forall\omega \le 0 \right\}\ . \tag{56}$$

Definition 2. Let $\psi \in L^2(\mathbb{R})$ be a fixed function, hereafter called the wavelet. The *continuous wavelet transform* (CWT) of any finite energy analytic signal $f \in H^2_+(\mathbb{R})$ is the function of two variables

$$T_f(b,a) = \langle f, \psi_{(b,a)}\rangle = \frac{1}{\sqrt{a}} \int f(t)\,\overline{\psi}\left(\frac{t-b}{a}\right)\,dt\ , \tag{57}$$

where the variables b and a run over $\mathbb{R}$ and $\mathbb{R}^+$ respectively, and the wavelets $\psi_{(b,a)}$ are shifted and scaled copies of the reference wavelet ψ, defined by

$$\psi_{(b,a)}(t) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{t-b}{a}\right)\ . \tag{58}$$

It may be shown (see e.g. Ref.[11]) that if the wavelet ψ is such that

$$0 < c_\psi = \int_0^\infty |\hat{\psi}(\omega)|^2\,\frac{d\omega}{\omega}\ < \infty \tag{59}$$

then for all the wavelet transform maps $H^2_+(\mathbb{R})$ into the space $L^2(\mathbb{R}\times\mathbb{R}^+, dbda/a^2)$, and that the transform is invertible on its range: for all $f\in H^2_+(\mathbb{R})$, one has

$$f = \frac{1}{c_\psi}\int T_f(b,a)\,\psi_{(b,a)}\,\frac{db\,da}{a^2}\ . \tag{60}$$

The continuous wavelet transform may be given a group-theoretical interpretation along the same lines as before. In particular, it may be completely characterized by its covariance properties, namely covariance with respect to translations and rescalings, which generate the so-called *affine group* G_{aff}.

4.2. The affine group and its representations

The *affine group* G_{aff} is the group generated by elements of the form $\{(b,a)\in\mathbb{R}\times\mathbb{R}^+\}$, with group multiplication

$$(b,a)(b',a') = (b+ab', aa')\ , \tag{61}$$

and may be realized as a matrix group, namely the group of matrices of the form

$$(b,a)\approx\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}\ .$$

The unit element is $(0,1)$, and the inverse of $(b,a)\in G_{aff}$ reads

$$(b,a)^{-1} = \left(-\frac{b}{a},\frac{1}{a}\right)\ . \tag{62}$$

Unlike the Weyl-Heisenberg group, the affine group is not unimodular. It possesses a measure μ which is left-invariant**, defined by

$$d\mu(b,a) = \frac{db}{a}\frac{da}{a}\ , \tag{63}$$

but this measure is not right-invariant. The right-invariant measure is given by

$$d\mu'(b,a) = db\,\frac{da}{a}\ , \tag{64}$$

The affine group acts on $H^2_+(\mathbb{R})$ by an irreducible unitary representation π: for all $f\in H^2_+(\mathbb{R})$ and $(b,a)\in G_{aff}$,

$$\pi(b,a)f(t) = \frac{1}{\sqrt{a}}\,f\left(\frac{t-b}{a}\right)\ . \tag{65}$$

The representation is square-integrable,[13,14] so that if $\psi\in H^2_+(\mathbb{R})$ is such that the condition (59) is fulfilled, then the map $T: f\in H^2_+(\mathbb{R})\longmapsto T_f$ defined by

$$T_f(b,a) = \langle f, \pi(b,a)\psi\rangle\ , \tag{66}$$

and identified with the CWT, defines an unitary map (after an appropriate normalization) between $H^2_+(\mathbb{R})$ and $L^2(G_{aff}, d\mu)$, as expressed by Equation (60).

The left regular action of G_{aff} on $L^2(G_{aff}, d\mu)$ is introduced as before (see (21)): if $F\in L^2(G_{aff}, d\mu)$, we write for $(b',a')\in G_{aff}$

$$[\lambda(b',a')F](b,a) = F\left(\frac{b-b'}{a'},\frac{a}{a'}\right)\ . \tag{67}$$

**in the sense that $d\mu((b',a')(b,a)) = d\mu(b,a)$ for all $(b',a')\in G_{aff}$.

This defines an unitary representation of G_{aff} on $L^2(G_{aff}, d\mu)$. The infinitesimal generators of λ are of interest: as before, introduce the differential operator $\check{\nabla} = (\check{\partial}_b, \check{\partial}_a)^t$, where for any smooth enough function F, we have set

$$\check{\partial}_b F(b,a) = \partial_b F(b,a) \tag{68}$$
$$\check{\partial}_a F(b,a) = a\partial_a F(b,a) + b\partial_b F(b,a) \ . \tag{69}$$

Using Taylor series expansions one may show that for all $F \in L^2(G_{aff}, d\mu)$,

$$[\lambda(v_b, v_a)F](b,a) = e^{-v_b\check{\partial}_b} e^{-\log(v_a)\check{\partial}_a} F(b,a) \ . \tag{70}$$

Similarly, the right-regular representation acts on $L^2(G_{aff}, d\mu')$, and is defined by

$$[\rho(b', a')F](b,a) = F\left(b + ab', aa'\right) \ . \tag{71}$$

The corresponding infinitesimal generators are given by $\widehat{\nabla} = (\hat{\partial}_b, \hat{\partial}_a)^t$, where for any smooth enough function F, we have set

$$\hat{\partial}_b F(b,a) = a\partial_b F(b,a) \tag{72}$$
$$\hat{\partial}_a F(b,a) = a\partial_a F(b,a) \ . \tag{73}$$

Then we have that for all $F \in L^2(\mathbb{R} \times \mathbb{R}^+)$,

$$[\rho(v_b, v_a)F](b,a) = e^{v_b\hat{\partial}_b} e^{\log(v_a)\hat{\partial}_a} F(b,a) \ . \tag{74}$$

Again, the left regular and right regular representations commute with each other, and with the infinitesimal generators of one another. As before, we denote by $\mathbb{P}$ the phase space which is now isomorphic to the affine group itself.

4.3. Constant phase deformations and reassignment

We now address the problem of deforming a CWT. We want to provide a geometrically natural prescription for wavelet reassignment maps. We are interested in finding a vector field

$$\underline{v} : (b,a) \to \underline{v}(b,a) = (v_b(b,a), v_a(b,a))^t \in \mathbb{R}^2$$

defined on the phase space, such that at first order in $\epsilon > 0$, the phase of the CWT remains unchanged under the infinitesimal (right) action of the affine group. For convenience, let us introduce the auxiliary vector

$$\underline{w} = \underline{w}(b,a) = (v_b(b,a), \log(v_a(b,a))) \ . \tag{75}$$

Let F be a smooth function defined on the phase space $\mathbb{P}$. For $\epsilon > 0$ and $\underline{v} \in \mathbb{R}^2$, let $\tau_{\underline{w}}(\epsilon) : L^2(\mathbb{P}) \to L^2(\mathbb{P})$ denote the operator defined by

$$\tau_{\underline{w}}(\epsilon)F(b,a) = e^{-\epsilon v_b\hat{\partial}_b} e^{-\epsilon\log(v_a)\hat{\partial}_a} F(b,a) \ . \tag{76}$$

At first order in ϵ, we have

$$\tau_{\underline{w}}(\epsilon)F(b,a) = F(b,a) - \epsilon\underline{w}(b,a) \cdot \widehat{\nabla}F(b,a) + O(\epsilon^2) \ . \tag{77}$$

Let now $f \in L^2(\mathbb{R})$, and let T_f denote its CWT. Let us write

$$T_f(b,a) = |T_f(b,a)|\, e^{i\Omega(b,a)} \ , \tag{78}$$

where $\Omega(b,a) = \arg(T_f(b,a))$. One immediately sees that, again at first order in ϵ,

$$\tau_{\underline{w}}(\epsilon)T_f(b,a) = T_f(b,a) - \epsilon T_f(b,a)\left(\underline{w}\cdot\frac{\nabla^m|T_f(b,a)|}{|T_f(b,a)|} + i\underline{w}\cdot\nabla^m\Omega(b,a)\right) + O(\epsilon^2)\ , \tag{79}$$

where, as above, we have introduced the *modified gradient* ∇^m, which in this case coincides with the infinitesimal generator $\widehat{\nabla}$:

$$\nabla^m\Omega(b,a) = \begin{pmatrix} \partial_b^m\Omega(b,a) \\ \partial_a^m\Omega(b,a) \end{pmatrix} = \begin{pmatrix} a\partial_b\Omega(b,a) \\ a\partial_a\Omega(b,a) \end{pmatrix} \tag{80}$$

Therefore, imposing that the argument of $T_f(b,a)$ be unchanged under the right action of the group imposes that the vector field $\underline{w}$ be of the form

$$\underline{w}(b,a) = \frac{\sigma}{2\pi}\begin{pmatrix} \partial_a^m\Omega(b,a) \\ -\partial_b^m\Omega(b,a) \end{pmatrix} = \frac{\sigma}{2\pi}\begin{pmatrix} a\partial_a\Omega(b,a) \\ -a\partial_b\Omega(b,a) \end{pmatrix}\ , \tag{81}$$

where $\sigma = \sigma(b,a) \in \mathbb{R}$ is a real number. Again, we obtain a form for the vector field which involves phase derivatives of the CWT.

The reassignment may be performed as before. Let us choose $\sigma = 1$ for the sake of simplicity. Given a vector field $\underline{w}(b,a)$ satisfying the condition (81), and $\epsilon > 0$, we have the following prescription

$$T_f(b,a) \longmapsto T_f\left(b - a\epsilon v_b(b,a), \frac{a}{v_a(b,a)^\epsilon}\right)\ , \tag{82}$$

We notice that such a prescription is quite different from the prescription studied in Ref.[4]. However, it is coherent with the structure of the affine group (i.e. additive on the first variable, and multiplicative on the second one).

5. CONCLUSIONS

We have presented in this article a geometrical setup for interpreting the methods based upon deformations of time-frequency representations. Starting from the description of the continuous wavelet and Gabor transforms in the framework of group representations, the time-frequency space appears as an homogeneous space with respect to the group action. The action of the group defines (generalized) translations on the time-frequency space.

Deforming a time-frequency representation (in this context, a square-integrable function on the time-frequency space) may be done using a local action of the group, i.e. by group-translating the time-frequency representation with translations which depend on the considered time-frequency location. At the infinitesimal level, such local translations are completely characterized by a vector field on the time-frequency space.

Two different group actions on the time-frequency space may be considered. We have shown that covariance requirements force one to limit to the so-called “right-translation”. The resulting deformations is then covariant with respect to the group action. For example, if the group under consideration is the affine group, the deformed wavelet representation of a shifted and scaled copy of a signal f equals a shifted and scaled copy of the deformed wavelet representation of the original signal f.

Once the group action has been chosen, the last question is the choice of the vector field, which yields the prescription for reassignment maps in reassignment algorithms. Such prescriptions have been proposed in the literature, but little is known when it comes to interpret them geometrically. We have shown here that the classical prescription[1] for CGT reassignment is obtained by imposing that the deformation preserves the phase of the transform. This remark also provides a complementary point of view on the recently introduced “differential reassignment” methods.[4] When applied to the case of the CWT, our approach also yields a prescription for deforming a continuous wavelet transform, which differs from the standard one. Applications of the corresponding wavelet reassignment are in progress, and will be presented elsewhere.

The work presented here raises several problems. To our opinion, the main one is that of inverting a reassigned time-frequency transform. It has been shown that such an inversion is possible[5] in a particular case. It is well known that continuous time-frequency transforms are extremely redundant, and that such a redundancy provides them with a great stability with respect to perturbations. One may conjecture that there exist deformations of time-frequency representations which preserve the inversion formula, and which possess the covariance properties described above.

Another interesting question concerns the extension of the results presented here to more general situations (higher dimensions, different geometries,...). We have deliberately chosen to emphasize the group theoretical point of view, because the latter provides a good framework for generalizations. For example, the results in the Weyl-Heisenberg case should generalize to higher dimensions without much difficulties, as higher dimensional Weyl-Heisenberg groups has the same structure as the one-dimensional. Generalization of the wavelet case may be done in several directions, as there exist several candidates for affine group in higher dimensions.

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